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New advances in the automated architectural space plan layout problem

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Abstract

Over the past four decades several computer models have been developed to solve what has come to be known as the space planning problem in architectural design. Although, space planning cannot be defined as an independent problem in the whole architectural design process, the problem is important if considered within the context of the design process itself, i.e. exploring the topological options of the design and not as an optimization or form finding exercise. Given this pretext, the space planning problem in architectural design becomes one of exploring and perhaps enumerating the various spatial arrangements possible. This line of thinking has been considered in the earlier years of the research, where the problem was typically formulated using graph theoretical approaches. At the core of these efforts was trying to answer the question "Can you practically enumerate all the possible space arrangements given relationships between space and other geometric constraints". However, this line of research has not received sufficient attention in the literature lately. Several findings related to graph theory have since been discovered in the last decade or so. This paper presents the new findings in graph theory that have direct implications on space layout and planning in architectural design. The paper presents the space planning problem from a graph theoretical approach with relevant definitions and posses questions related to the topology of the planar graphs that represent the spatial layout of any design. A refined definition of architectural floor plans as "simple, connected, labelled, planar graphs" is presented along with justification and discussions. Recent findings and algorithms related to enumerating rectangular dissections, and planar embeddings of planar graphs including area universal planar graphs are presented as well as the necessary and sufficient conditions for a graph to have a rectangular dual. The paper should be of interest to architectural professionals as well as researchers in the area.

Keywords: space layout, planning, graph theory, architectural design, algorithmic architecture

1 Introduction and background

Somewhere within the design process, architects find themselves solving layout configuration problems; trying to layout a number of spaces with certain topological relationships between them. It has been shown earlier in the literature that such layout problems, under certain conditions and constraints, can have a finite, albeit large, set of solutions (Cross 1977, Harary, et al 1978, Krishnamurti 1978, Baybars and Eastman 1980, Steadman 1983, Rinsma 1987, Mitchell, 1990, Hillier 1998 to name a few). On the other hand, architects often do not acknowledge such limitations always arguing that design possibilities are often infinite. Paradoxically, this argument is partially correct.

The quandary is that design problems are rarely well defined or structured. This makes the assertion of a finite solution space improvable. Somewhere within the architectural design process however certain problems arise that can be well structured and formulated so that a finite set of solutions are possible and indeed the possible solutions can be enumerated. However architects usually are not able to properly and efficiently formulate such problems mathematically. This is due to 3 main reasons: firstly, the nature of the architect-design relationship is such that the constraints, the objectives and the variables are fuzzy. Secondly, the rigorous and efficient mathematical formulation of the design problem may be time consuming, tedious and require a level of mathematical knowledge that architects usually do not posses. Thirdly, architects can often reach “near-optimum” solutions without the need to rigorously define the problem.

Nevertheless, at some functional level, architects often ask themselves when designing, did I consider all possible other alternatives? Is there a better alternative design? Obviously the word ‘better’ is very loose. But even more, the word ‘alternative’ is also ill-defined. Alternative can mean different form, color, etc…. On one level however it can mean an ‘alternative’ relation between the spaces in the building. Questions on how to enumerate or find some solutions that meet certain constraints have been tackled as early as 1960s (Levin 1967, Whitehead and Eldars 1964 for example). In this paper we show that it is still an open question and shed light on how recent advances in graph and combinatorial theory have made the solution more realizable. From the literature review there seems to be two separate approaches to solving the layout problem; one that is based on the facility design problem and tries to optimize the location of spaces (see for some examples of this approach). This approach borrows heavily from the facility layout problem in operations research and industrial engineering applications.

The second approach is based on studying the morphology and topology of the layout and is either based on dissecting rectangles to reach exhaustively all possible solutions (see Jo and Gero 1998 and Yoon 1992 for some examples of this approach) or is based solely on the graph theoretic approach which uses theorems and properties of graphs to either enumerate solutions or find solutions that meet certain configuration criteria (see Krishnamurti 1978 and Steadman 1983 for some examples of this approach). The dissection approach which was suggested earlier in the literature is often based on graph theoretic approaches as well. Our focus in this paper will be purely on the second approach. Although significant technical accomplishments were earlier on, these efforts could not be translated into practical design tools/software that could be used practically by professional architects. Several recent advancements in graph theoretic approaches can have impact on coming closer to a comprehensive solution to the problem. In the next section we present graph theoretic approaches.

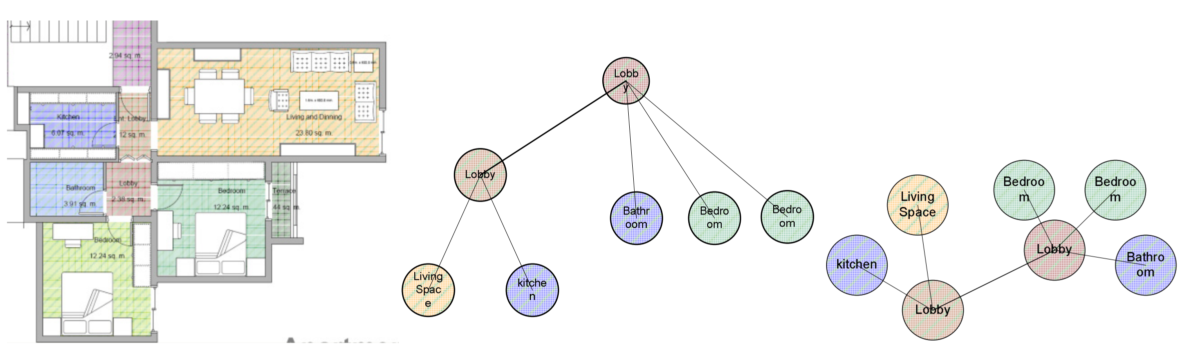

Figure 1. A floor plan and 2 tree embeddings of its graph dual

2 A precise definition of the architectural plan as a graph

Graph theory is the study of pair-wise relations between objects from a certain collection. Let us start by considering the architectural plan as a graph. A graph is simply a group of nodes (vertices) and links between. In its simplest form the nodes represent rooms and the links represents connectivity between the rooms (figure 1). Connectivity here can mean a door way but more importantly adjacency. Recently a new concept of adjacency was defined; contact graphs (Epstein 2006 for example). Contact graphs define that some sort of contact has to exist between the two spaces which means that voids or gaps can be incorporated between the spaces. This new definition allows for more varieties to be studied as will be defined below. In any case, a graph may have multiple embeddings in the plan, i.e. it can be drawn in different ways. Several graph models of building space have been developed (see Franz et al 2005 for a brief review).

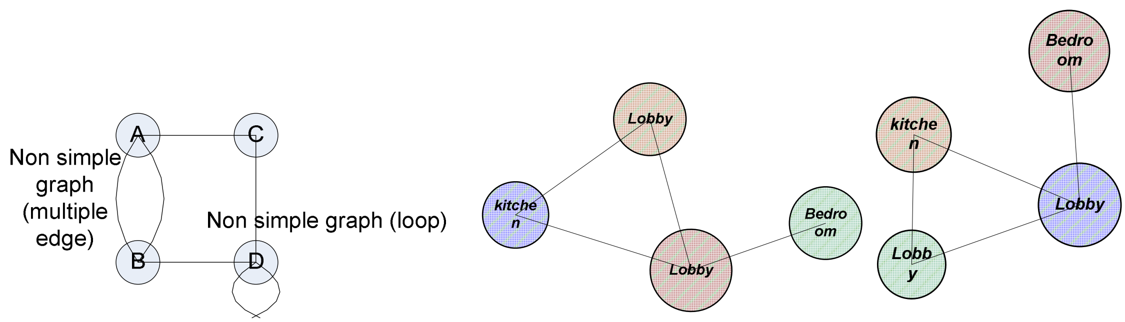

Figure 2. Simple graphs and two isomorphic embeddings a floor plan graph

Some graphs are an actually trees such as the one shown in figure 1 which means that all their embeddings are topologically equivalent. In architectural terms, this means the geometric relation (above, next to, under, etc…) between the spaces can only take one form. Some important properties of architectural plans graphs include that they are planar, undirected, simple, connected, labeled, and sometimes it takes the form of a tree. The most important feature of a graph representation of an architectural configuration is planarity, which means that it can be drawn in the plane without edges intersection each other. The faces of planer graphs are bounded by three edge, and v-e+f=2 (v is the number of vertices, e is the number of edges and f is the number of faces). Of course the prerequisite of graph planarity would be true only if we are thinking in 2 dimensions, but non-planar graphs can still be realizable as plans if we consider the third dimension and varying levels. Finding 3D or lattice embeddings of non-planar graphs in architecure is a completely new line of research that has not been studied thus far though.

However for the purpose of this paper we limit ourselves to the 2-dimensional case. An architectural graph is also connected since there has to be a path from any point to any other point in the graph (it does not matter if room A opens onto room B or vice versa, since it is still the same). The concept of maximal planar and maximal connected is also important since they mean that the addition of one more relations (edge between) any two space renders the graph non-planar (and therefore not realizable as a plan) or the removal of one edge makes the graph unconnected. An architectural graph is undirected (although Grason 1971 suggests a model which is directed and Flemming 1992 also uses directed graphs but these are used for algorithmic reasons and have no representational value) since the direction of the relationship between the building's spaces is bi-directional. Remember also that an adjacency matrix is triangular (meaning that the lower part of the matrix has to be the same as the one above (in graph terminology no multiple edges are allowed) and also that the diagonals must be zeros (in graph terminology, no loops). This kind of adjacency matrix leads to what is known as a simple graph (figure 2 a). Finally, the question whether architectural graphs are labeled relates to isomorphism of the graph. Two graphs are isomorphic for some ordering of their vertices their adjacency matrices are equal. This means that architectural plans may have the same structure if one ignores individual distinctions of the spaces, Figure 2b. Whenever individuality of the spaces of the plan is important for correct representation of the design the plan is modeled by labeled graphs (although in some cases digraphs, colored graphs, rooted trees can be used). The computational problem of determining whether two finite graphs are isomorphic is called the graph isomorphism problem. We know that when hen the graphs are counted up to isomorphisms, i.e., we consider unlabeled planar graphs, the situation gets even more complex (Bodirsky et al 2007b). In the next section we present some of the recent findings relating to graph theoretic approached to the architectural layout problem.

3 Advances in the graph theoretic approach and implications to the architectural layout problem

In this section we consider the advances in graph theory and their implications on the architectural design problem in three main areas; graph enumeration, the existence of a rectangular dual and dissection.

3.1 Graph enumeration

Graph enumeration has advanced significantly during the last 2 decades (although the architectural layout research has not for the most part utilized these findings). Graph enumeration research answers such questions relating to architectural design such as: "given a set of rooms how many different adjacency matrices can we have" (Incidentally, for a general graph the answer is equal to the number of permutations of n elements, which is *n!*, and for an undirected labeled graphs it is $2^{\binom{n}{2}}$). The question is architecture ars combinatoria was suggested earlier by Hillier (Hiller 1998) as "… *(if) we must conclude that buildings as a combinatorial system take the form of one combinatorial explosion within another with neither being usefully countable except under the imposition of highly artificial constraints,…how then should we account for the fact that there do seem to be rather few basic ways of ordering space in buildings"*. We suggest that the right question has to be asked when it comes to combinatorial nature of architectural design. In architectural design, as was stated earlier, we are generally interested in only the connected graphs (since no room can remain unconnected in a typical architectural plan). In addition to the condition of being connected, we have to consider the question of labeling. This means that will we consider the actual space use when we are enumerating the different arrangements or not? Sloane and Plouffe 1995 have shown that the number of n-node connected unlabeled graphs for with n =1 , 2, ... nodes are 1, 1, 2, 6, 21, 112, 853, 11117, 261080, ... The total number of (not necessarily connected) unlabeled n-node graphs is given by the Euler transform of the preceding sequence, 1, 2, 4, 11, 34, 156, 1044, 12346, ... King and Palmer 1998 have calculated it up to n= 24 (and it was found to be an astronomically large number). On the other hand, the numbers of connected labeled graphs on n-nodes are 1, 1, 4, 38, 728, 26704, ...and the total number of (not necessarily connected) labeled n-node graphs is given by the exponential transform of the preceding sequence: 1, 2, 8, 64, 1024, 32768, ... (Plouffe 1995, p. 20).

Another aspect of architectural graphs that adds to the complexity of the question is the condition of planarity. Wilson ... (Wilson 1975) have shown that the number of planar graphs with n=, 2, ... nodes are 1, 2, 4, 11, 33, 142, 822, 6966, 79853, etc… But this number is for all planar graphs and not necessarily, connected or labeled.

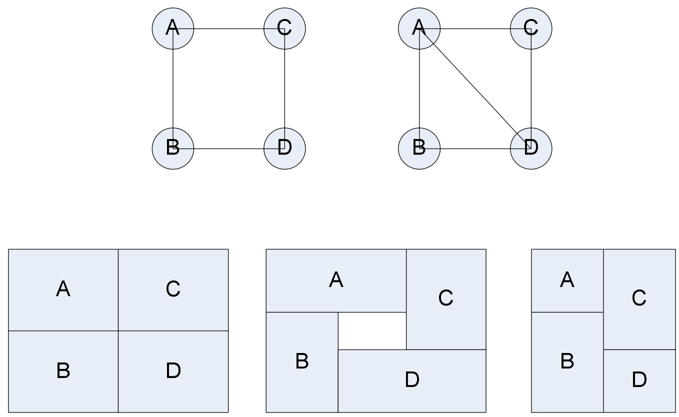

Figure 3. *a)* simple graphs, *b)* isomorphic graphs, *c)* the floor plan *d)* and 2 tree embeddings of its graph dual

So the question now becomes, "*How many simple undirected, labeled (and non-isomorphic) connected, simple, planar graphs of node n?*" Although this question is still open, several related issue have been addressed recently. Bodirsky et al 2007 showed that for general graphs the number of labeled and the number of unlabeled graphs are asymptotically equal, since almost all graphs are asymmetric. For planar graphs: the number of labeled graphs is much larger than the number of unlabeled graphs, since almost all planar graphs have a large automorphisms group. This is important for our problem because it shows that the solution space is actually smaller than what was discussed in before in the architecture related literature. Bodirsky et al 2007a enumerated labelled planar graphs and derived recurrence formulas that count all such graphs with n vertices and m edges. This enumeration does not take into consideration the connectedness or that the graphs have to be simple however and therefore the values derived from the developed recurrence formulas would yield results that are larger than expected for the architectural layout problem. Bodirsky et al 2007b also enumerated connected graphs but only cubic graphs were considered (planar graphs where every vertex has exactly three neighbors). Gimenz and Noy 2009 recently surveyed the literature on generating planar graphs and this survey shows that an exact answer to the question defined above is still open.

3.2 The Rectangular Dual

The second advancement in Graph theory relates to the existence of a Rectangular Dual to the planar graph. Note that in the above section we considered planar graphs as combinatorial objects, regardless of how many different embeddings they may have in the plane. Informally, a planar embedding of a graph is a way to draw the graph in the plane and since some planar graphs can be drawn in different ways in the plane, they may have multiple embeddings. The advances on this front during the last two decades are three fold; first development of algorithms for testing the planarity of graphs, second development of algorithms to check the existence of a rectangular dual and third, development of algorithms that generate rectangular duals with certain properties.

For a long time there were a number of efficient algorithms for planarity testing, which were unfortunately all difficult to implement. Checking the planarity of architectural graphs is important because non-planar graphs are not realizable as floor plans and thus it is crucial to check the adjacency matrices input by the user to verify that they are realizable in the first place. This point was not considered rigorously in the early research in the field, since no rigorously proven efficient algorithms were present then. Therefore checking for planarity can act as a semaphore so that no search for a solution starts until the planarity has been verified. Designers will often specify several desired relations between the spaces and this may render the final architectural graph non-planar. On this front, linear time planarity testing algorithms have previously been proposed such as those by Hopcroft and Tarjan 1983, and by Booth and Lueker 1976. However, their approaches are quite involved. A simple linear time testing algorithm based only on a depth-first search tree was recently presented in Habib and Paul 2005. In that algorithm a graph-reduction technique is adopted so that the embeddings for the planar bi-connected components constructed at each of the iterations never have to be changed.

Secondly, even if planarity of a graph is guaranteed, not every plan graph however has a rectangular dual. Rectangular duals have been studied extensively, including much work on characterizing graphs that admit rectangular duals, transforming those that do not by adding new vertices, and constructing rectangular duals in linear time. In the architectural literature no mention was made of the formal conditions on characterizing planar graphs was given although KOZM84a at earlier proved that if G is planar and every face (except the exterior) is a triangle, all internal vertices have degree larger than or equal to 4 and, all cycles that are not faces have length larger than or equal to 4, then a rectangular dual does exist. Using these necessary and sufficient conditions, Kozminski and Kinnen 1984 were able to obtain a rectangular dual if one existed. More recently Bhasker J. and S.Sahni developed a faster and more efficient algorithm to determine if a planar graph G satisfies conditions above. In addition, Jayaram Bhasker and Sartaj Sahni developed a linear time algorithm to determine if a given planar triangulated graph has a rectangular dual works only with maximal planar graphs. This is important for architectural purposes because it allows architects to check for the feasibility of their constraints.

Thirdly and perhaps of most interest to the architectural problem is the relatively recent development of a number of algorithms that can generate rectangular duals quickly, efficiently as well as algorithms that can generate duals meeting certain prescribed constraints. These constraints could resemble issues such as orientation for example. Marwan A. Jabri 2007 presented an efficient algorithm that transforms an arbitrary connected graph, representing an integrated circuit, into another graph that is guaranteed to fulfill these conditions and to admit rectangular duals. The algorithm is conceptually simpler than the previous known algorithm. The coordinates of the rectangular dual constructed by our algorithm are integers and have pure combinatorial meaning. This advancement is of interest to architecture plans since it can actually change the topology of a plan so that it can be physically realized and will be useful in case a designer over constrains the plans in some way for example. Furthermore (Eppstein and Mumford 2009) developed an algorithm that can check for the existence of a rectangular dual that satisfies certain constraints on the orientations of the adjacencies of its regions; such constraints may be particularly relevant for cartographic applications of these layouts.

The related concept of rectangular layouts has also received attention recently. A rectangular Layout is similar to a rectangular dual, with the exception that voids or gaps are allowed between the rectangles. So unlike a rectangular dual that must form a dissection of its enclosing rectangle; i.e., it allows no gaps between rectangles, rectangular layouts allow this. The prohibition of gaps in rectangular duals limits the class of graphs that admit rectangular duals; Rinsma 1987 found that paths are the only trees that have such duals. In general, any planar graph admitting a rectangular dual must be internally triangulated. This means that another space has to be created or in more practical terms the shape of the space no longer will remain rectangular. However the restriction that each required internal triangulation does not apply to layouts, since gaps are allowed which gives many advantages to layouts over duals. The cleaner, less specified definition of layouts characterizes a class of graphs that is both more general (including all trees, for example) and also much simpler to formalize. In addition (Hi 2007) showed that a small graph might require a significantly larger dual than a larger graph. A very interesting recent finding in this area relates to Area-Universal Rectangular Layouts. A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout. (Eppstein etal 2009) identified a rectangular layout is area-universal if and only if it is one-sided (i.e. have a space with anyone of its sides spanning the entire width of the plan) and they also showed how to find an area-universal layout for a given set of adjacency requirements whenever such a layout exists. As for rectangular duals for directed graphs (which rarely are used to represent architectural floor plans), it was recently found that a dag G has a directed rectangular dual if and only if G is a planar all interior faces are triangles and all interior vertices have in-degree and out-degree and that the boundary of E is embeddable.

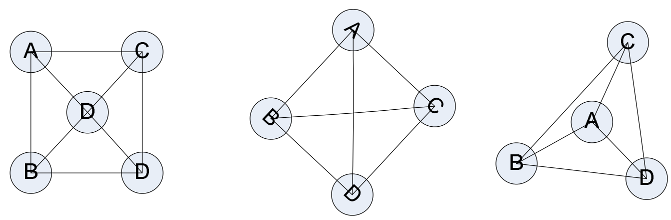

Figure 4. a triangulated planar graph and a planar graph with a two embeddings

Note that while graph planarity is an inherent property of a graph, it is still in general possible to draw non-planar embeddings. For example, the two embeddings above both correspond to the planar tetrahedral graph, but while the left embedding is planar, the right embedding is not. Earlier on Robinson and Janjic 1983 showed that, if areas are specified for rooms with a given maximal outer-planar adjacency graph, then any convex polygon with the correct area can be divided into convex rooms to satisfy both area and adjacency requirements. If the perimeter and rooms must be rectangular, un-dimensioned floor plans can be found to fit any maximal outer-planar adjacency graph with at most four vertices of degree 2. It is shown that in some cases it is not always possible to satisfy the area constraints. Another important finding relates to finding rectangular dual of non-planar graphs using shapes other than rectangles. It was shown in Wei-Kuan Shih and Wen-Lian Hsu 2006 that L- and T-shapes in addition to rectangles are always sufficient to represent a planar graph. A rectangular dual is not necessarily unique. Also, Kurowski 2003 developed a simple algorithm for computing a floor-plan of a given plane near-triangulation using modules which are the union of two rectangles and are T-, L- or I-shaped. The number of T-shaped modules is at most $\frac{1}{2}(n - 2)$, all T-shaped modules are uniformly directed, the size of the picture is at most $n \times n - 1$. This kind of arrangement is new to the architectural literature and was briefly discussed in the early years as 'polymino arrangements' (Steadman 1983) but the new algorithm and theorems open a new door for research possibilities in architecture.

Figure 5. A planar graph with no rectangular embedding and a L- shaped emebedding

3.3 Dissection

As recent as 2007 Hillier writes "*for values of n (the number of cells in a rectangular "dissection") much greater than 10, the extent of combinatorial variety becomes so great that a complete enumeration is of little practical purpose; and indeed that for values of n not much larger than this, enumeration itself becomes a practical impossibility*". This earlier line of research has received the least attention in the recent years. The idea behind dissection is that some architectural floor plans are sliceable. A floor plan is sliceable if it can be recursively deconstructed by vertical and horizontal lines extending fully across the bounding box. Minimizing the area of non-sliceable floorplans is NP-hard under various constraints, while the area minimization of sliceable floorplans is tractable (GANSNER et al 2008). Not all floor plans can be realized by sliceable equivalents though. Therefore some researchers are currently working on identifying and generating sliceable floor plans where possible as well as minimizing the area of non-sliceable floor plans by various heuristics [(GANSNER et al 2008). Shin-ichi NAKANO 2003 recently developed a simple algorithm to generate all based floor plans with at most n faces. He also was able to generate all based floor plans with exactly n faces containing at least k1 and at most k2 inner rooms. This finding is important because it we now can control issues such as view. An important open question though is whether rectangular duals and layouts can cover a wider range of solutions than sliceable floor plans.

4 Conclusions

So why do architects have to know about things like graph theory, the museum guard problem or rectangular dissections. The rational is very similar to why engineers have to learn about Laplace transforms, Bessel functions and other mathematical techniques. Engineers do not have to actually develop mathematical techniques but instead use them to understand, analyze and solve the problems at hand. Similarly, architects have to know the implications of relevant techniques on their design. It is not anticipated that they develop complex proofs of theorems or evaluate algorithm complexity but rather know how to use these algorithms to analyze and develop their designs as well as know the implications of theorems on the possibilities of design.

The above findings can lead to a tool that could enumerate all the possible rectangular duals for a building plan graph. Although this seems to have been answered in previous research, the problem is still open if we consider layouts from planar, connected, non-isomophic simple graphs. The process would be first to see how many different connected, simple, planar, labeled graphs one can draw for any given design with n spaces. Then for each of those graphs find all the possible embeddings. For each embedding check to see if there is a rectangular dual (here we are talking about rectangular duals of the graphs but perhaps due to the boundary of the sites we need to consider 'polygonal embeddings'), and then finally select those that fit the dimensional constraints set by the designer. This approach will generate a true exhaustive set of possible designs. It remains to be seen though, that after correctly specifying and constraining the problem, a manageable set of alternatives will be generated or not? Other directions for future research include increasing the speed of algorithms and more importantly considering the third dimension.

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