Modelling the nonlinear behaviour of a cracked reinforced concrete beam

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Abstract

This paper presents a model of crack formation and propagation in a reinforced concrete beam section. The model is aimed at understanding the nonlinear behaviour of cracked RC beams at residual deformation, which is important for structural health monitoring. The loading moment-rotation curve is developed using a fictitious crack model and Cornelissen’s constitutive relationship for concrete. For unloading, a new bi-linear constitutive relationship is applied, in which the initial stiffness is defined by a focal point in the elastic compression region. The material is then assumed to achieve its initial modulus of elasticity either when the crack closes or the stress becomes compressive. The degree of nonlinearity at residual rotation is quantified by the curvature of the unloading moment-rotation curve at zero moment. The sensitivity of the degree of nonlinearity to damage level, crack height and focal point location is investigated. It is found that the nonlinearity is sensitive to both the location of the focal point and the level of damage, but the trend of changes in the nonlinearity with damage levels is similar for each focal point. It is also observed that the nonlinearity increases with the level of damage up to about 30% of the failure load and then decreases with further damage, which is consistent with previously reported experimental data.

Keywords: structural health monitoring, fictitious crack, nonlinear stiffness, RC beams

1 Introduction

In recent years, the Structural Health Monitoring (SHM) of civil infrastructure has attracted great interest. Researchers have been searching for an effective and reliable process of acquiring, managing and interpreting structural performance data to assist in damage detection and asset management. Much of the work has focused on vibration-based techniques, because the vibration characteristics can easily be acquired and provide global information on the structural condition. However, despite over 20 years of study, many significant challenges remain. Importantly, cracked reinforced concrete (RC) structures have been found to exhibit significant residual nonlinearity (Eccles et al., 1999; Neild et al., 2002), which has been attributed to the transition from crack open to crack closed during the vibration cycle (Owen et al., 2002). This nonlinearity means that measured values of modal properties are amplitude dependent which makes simple SHM procedures based on changes in natural frequency or mode shape invalid.

However, in the last ten years, researchers have considered exploiting this nonlinear behaviour of damaged RC structures as a tool for SHM (Van Den Abeele and De Visscher, 2000; Peng et al., 2008). The degree of nonlinearity appears to vary consistently with damage, and so if an appropriate
means of paramatising the nonlinearity can be found, this will be useful in the further development of SHM. Other researchers have also attempted to model the vibration behaviour of damaged RC beams using either breathing crack models (Chondros et al., 2001) or nonlinear crack models (Tan, 2003). These trials have managed to replicate the nonlinear phenomenon, but failed to reproduce the experimental behaviour quantitatively. This is believed to be due to the lack of clear understanding of both the formation mechanism and residual vibration behaviour of cracks in RC beams.

In this paper, a model is developed for the crack initiation and growth in a RC beam section (Figure 1). The model also accounts for the unloading behaviour to capture the residual nonlinearity that is important for SHM of concrete structures. The influence of the assumed constitutive relationship for unloading on the degree of nonlinearity predicted is investigated. Comparisons are then made with experimental measurements of the degree of nonlinearity. Table 1 shows the material properties used in the modelling.

![Figure 1, Section’s dimensions and reinforcement](image)

<table>
<thead>
<tr>
<th>Material</th>
<th>Compressive strength (N/mm²)</th>
<th>Tensile strength (N/mm²)</th>
<th>Modulus of elasticity (kN/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>30</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>Steel</td>
<td>-</td>
<td>250 (Top), 460 (Bottom)</td>
<td>200</td>
</tr>
</tbody>
</table>

2 Modelling strategy

The fictitious crack (FC) approach is adopted in this study, in which the cracks are described using the stress against crack width relationship. The crack forms when the ultimate tensile stress of concrete ($f_t$) is reached. The effect of the crack spreads over a layer of width ($h_c$), known as the equivalent elastic layer. The width of the equivalent elastic layer is an essential parameter of the model as it represents the stiffness of the section. It is taken as half the overall depth of the section ($h_c = 0.5h$) according to a previous study by (Ulfkjaer et al., 1995).

The material points on the crack formation path are presumed to be in one of three possible states (Figure 2). These are: (1) linear-elastic state before crack formation, (2) a fracture state where material softened due to the cohesive forces in the fracture process zone, and (3) a state of zero stress when the crack width is beyond the critical crack width ($w_c = 160 \mu m$) (Ulfkjaer et al., 1995).

![Figure 2, Stress distribution in phases of fracture process in concrete beams (Ulfkjaer et al., 1995)](image)
2.1 Assumptions for the modelling process

The assumptions below were followed in developing the crack model:

- A flexural crack is assumed to develop in the tension zone.
- The strain softening behaviour of concrete in tension is assumed to play the dominant role in causing the nonlinear behaviour. Other factors such as steel-concrete bond and interface behaviour are neglected as they have little effect on the nonlinearity (Neild et al., 2002).
- Only the uniaxial behaviour of plain concrete is considered and the effect of confinement is neglected.
- Only tensile and compressive reinforcement is considered.
- Plane sections are assumed to remain plane.

2.2 Stress-strain/crack width relationship for concrete

In this study, the constitutive relationship for concrete is assumed to be dependent on the level of damage and whether the section is being loaded or unloaded. When load is applied, concrete is modelled as a linear-elastic material in compression and also in tension up to the ultimate tensile stress, which is taken as one tenth of the compressive strength ($f_{cu}$). After the ultimate tensile stress is reached, the crack initiates and the stress starts to decay with increasing deformation until it reaches zero at the critical crack width. The stress-crack width relationship is modelled using the stress-deformation envelope curve developed by (Cornelissen et al., 1986), which is described by (Eq. 1).

$$f_c = f_t \left[ 1 + \left( C_1 \left( \frac{v_c - v_{ct}}{w_c} \right) \right)^3 \right] e^{-C_2 \left( \frac{v_c - v_{ct}}{w_c} \right)} \left( 1 + C_1 \left( \frac{v_c - v_{ct}}{w_c} \right)^3 \right) e^{-C_2}$$

(1)

Where $f_c$ is concrete stress in tension zone, $v_c$ is deformation of concrete, $v_{ct}$ is critical deformation of concrete, and $C_1$ and $C_2$ are empirical values equal 3 and 6.93 respectively.

The stress-strain relationship for compressive and linear-elastic tensile regions is combined with the stress-crack width relationship into one single stress-deformation ($f_c$-$v_c$) relationship (Figure 3(a)). On the ascending branch, the stress varies linearly with the elongation as there is no crack (Eq. 2).

$$f_c = \frac{v_c}{h_c E_c}$$

(2)

On the descending branch, after crack initiation, the total elongation consists of the linear elastic elongation plus the crack width (Eq. 3), where the tensile stress of concrete is described by (Eq. 1).

$$v_c = \frac{f_c h_c}{E_c} + w$$

(3)

Figure 3, (a) Loading stress-elongation relationship for concrete (b) Unloading stress-elongation relationship for concrete in tension.
The procedure followed in unloading is based on two main assumptions. First, points in the compression region and those in the linear-elastic zone of the tension region are unloaded linearly. Second, points in the descending branch of the envelope are unloaded bi-linearly. The initial unloading path is linear, defined by focal point in the elastic compression region (Figure 3(b)). However, when either the stresses become compressive or the elongation reduces to the critical value, the crack is assumed to close and the gradient of the subsequent stress-elongation line is assumed to be the modulus of elasticity of the concrete.

2.3 Stress-strain relationship for steel reinforcement

A linear stress-strain relationship is assumed for the steel reinforcement for both loading and unloading. This assumption is valid as the purpose of the model is to scrutinise the nonlinear behaviour of RC beams before failure and so the stress in the reinforcement is always less than the yield stress of the steel reinforcement. The strain in the steel reinforcement of the crack model is also described in terms of the elongation (Eq. 4).

\[ v_s = \frac{f_s h_c}{E_s} \]  

(4)

Where \( v_s \) is elongation in the steel and \( f_s \) is the stress in steel reinforcement.

2.4 Crack modelling steps

The depth of the section is divided into a number of strips and an elongation is applied at the bottom fibre of the section. An initial depth to the neutral axis is assumed and stress distribution is calculated assuming plane sections remain plane. Then, the stresses are integrated to find the total force of the section. Another depth to the neutral axis is assumed and the total force of the section is again calculated following the same steps. Newton-Raphson iterative solver is used to adjust the position of the neutral axis and ensure equilibrium.

The bending moment resistance of the section is calculated by integrating the force in each strip. The rotation of the interface is found from the applied elongation using beam bending model. This procedure can also be used to model multiple cracks in a beam by relating the elongation in each section of the beam to the section interface rotation. The moment-rotation curve is developed by either increasing (loading path) or decreasing (unloading path) the applied elongation up to the required level. The crack model is solved numerically using Matlab.

In experiments on cracked RC beams to investigate the use of vibration data for SHM, nonlinearity was observed after damaging loads had been removed and for small displacements associated with vibration measurement. Hence, it was necessary to determine the nonlinear behaviour at the residual deformation. This was found by fitting a cubic polynomial to the moment-rotation curve as the section was unloaded.

3 Results

Figure 4(a) shows the moment-rotation curve developed by the RC crack model. The moment-rotation relationship is linear up to the cracking moment (~ 3 kNm). Beyond this point the relationship is not linear as the FC forms and the contribution of concrete under the neutral axis reduces. Thereafter, the contribution of concrete becomes negligible and the tensile resistance of the section is only provided by the steel reinforcement. Figure 4(b) illustrates the gradient of the moment-rotation curve. It is clear that the stiffness is initially constant and then drops sharply at the crack formation phase. After the crack propagates and the contribution of concrete under the neutral axis reduces, the stiffness increases and tends to behave linearly with increasing deformation.
Different ten unloading paths are developed by varying the position of the focal point between a point in the compressive zone with a value of stress ($f_f$) and another point with a value of stress ($-0.1f_f$). The unloading is continued to a negative moment equals about 5% of the ultimate capacity of the beam (~18 kNm). This value is arbitrarily chosen so as to be larger than the amplitude of the vibration excitation when the beam is examined experimentally in future studies.

Figure 5(a-b) illustrates different unloading paths from two levels of damage (DL) (~25% & ~50%) and the nonlinearity at the residual deformation for each unloading path. It is clear that the nonlinear coefficient of the unloading paths from the first level of damage is increasing with the normalised focal point. The largest nonlinear coefficient is corresponding to the unloading path with largest residual rotation. Contrary to this, the nonlinear coefficient of the unloading paths from the second level of damage is decreasing with the normalised focal point. The largest nonlinear coefficient in this case corresponds to the unloading path with the lowest residual rotation. It is also interesting to note that the nonlinear coefficients of the unloading paths from the first damage level are low with little variation compared with those from the second level of damage.
The nonlinearity of the section for different levels of damage is then investigated to determine whether the nonlinearity is increasing or decreasing with the level of damage. Different unloading paths, from (25-80) % of the failure load, are developed by unloading to two normalised focal points (0.3 & 0.7), and the nonlinearity at residual rotation of each unloading path is compared with the damage level and the crack height from the bottom fibre of the section (Figure 6(a-b)). It can clearly be seen that the nonlinearity is increasing with the level of damage up to approximately 30% of the failure load and then starts to decrease. At the highest nonlinear coefficient the crack has grown up to around half the depth of the section (105 mm). Beyond that point, the nonlinearity decreases rapidly indicating that the cracked parts of concrete are unloaded linearly leading the system to exhibit a rather linear behaviour. This is because the crack width of the cracked parts of concrete is beyond the critical crack width and hence stresses of these parts change directly to compression. It can also be observed that the nonlinearity at 80% of the failure load is equivalent to that at 25% of the failure load. The nonlinear coefficients of the points unloaded to the first normalised focal point (0.3) are greater than those unloaded to the second focal point. However, the trend of the nonlinearity of both paths is comparable. This indicates that the nonlinearity is governed by both the position of the focal point and the level of damage. Similar findings have been detected experimentally by (Neild et al., 2002) who found that the nonlinear behaviour increases with damage up to 27% of failure load and then decreases in a reverse trend up to 91% of failure load.

![Figure 6](image.png)

**Figure 6.** (a) Changes in nonlinearity at residual rotation with damage level (b) Changes in nonlinearity at residual rotation with crack height

### 4 Conclusions

A model of a RC crack is developed to examine the nonlinear behaviour of cracked RC beams. The model studies the degree of nonlinearity of the unloading moment-rotation curves at residual deformation. It predicts that the nonlinearity increases with the damage level up to around 30% of the collapse load and then decreases with further damage. The nonlinearity is found to be at its highest when the crack height is half the depth of the section. The magnitude of the nonlinearity is also found to be sensitive to both the position of the focal point and the level of damage. The changes in nonlinearity with damage level and crack height have a similar trend for each focal point, and this trend matched previous experimental results. The next step in this work will be to determine the correct position of the focal point from empirical data. When this has been achieved, more detailed
study of the dependence of nonlinearity on damage level will be required so that the nonlinear parameter can be used as part of a SHM system based upon nonlinear vibration measurements.

References


